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ORIGINAL PAPER

Solvability of first order functional differential operators

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Abstract By using the methods of operator theory, all boundedly solvable extensions of minimal operator generated by first order functional differential-operator expression in the Hilbert space of vector-functions on finite interval have been described. The operator framework is also applied to the study of structure of spectrums of these extensions. Applications of obtained results to the concrete models are illustrated.

Keywords Functional differential-operator expression · Boundedly solvable extension · Spectrum · Resolvent operator

Mathematics Subject Classification 47A10 · 47A20 · 34K06

1 Introduction

It is known that the equation $\frac{dx}{dt} = F(x),\ x = x(t)$ with an operator F defined on a set of absolutely continuous functions is called the functional differential equation (see [1]). Many problems arising in biology, economy, control theory, electrodynamics, chemistry, ecology, epidemiology, tumor growth, neural networks and etc. is reduced the study of boundary value problems for functional differential equations for first and second order in different functional spaces.

To the memory of eminent acad. F.G. Maksudov (1930–2000).

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The quantitative and qualitative theory of such equations have been studied in many works (for example, see [1–6]). Functional differential equations arises in many areas of science and technology. Solvability of such equations and connected problems it is indisputable. It is known that many of mathematical problems in chemistry are expressed by functional differential equations with auxiliary boundary conditions. The problem in chemical reactions [7–9], chemical kinetics [10–13], in modelling of problems in theory of genetic regulation [14] and etc. can be expressed by time-delay functional differential equations. For example, the effect of a controlled delay on a chemical oscillator is described by functional differential equations (see [8]). The functional differential equations arising from the assumed mechanism together with flow terms are in form

$$\dot{C} = f(C) + k_0(C_0 - C),$$

where $f(\cdot)$ is the kinetic rate law, the C_0 is the in-flow concentration and k_0 is the flow rate constant. Numerically solutions of above differential equation have been investigated in [7].

Note that many fundamental laws of chemistry can be formulated by ordinary and functional differential equations.

In this work, the operator theory framework, the boundedly solvability and spectrum structure for wide class functional differential equations for first order in the Hilbert space of vector-functions on finite interval. For example, such investigations for the some retarded-type functional differential equations have been done in works [15–19].

The first work in area of extension of linear densely defined operator in a Hilbert space belongs to von Neumann. In his paper [20] all the selfadjoint extensions of the linear densely defined having equal and nonzero deficiency indexes symmetric operator in any Hilbert space have been described. But in 1949 and 1952 Vishik the boundedly (compact, regular and normal) invertible extensions of any unbounded linear operator in a Hilbert space have been established in works [21,22] and these results by Otelbayev, Kokebaev and Shynybekov have been generalized to the nonlinear operators and complete additive Hausdorff topological spaces in abstract terms in works [23,24]. Dezin [25] give a general methods for the description of regular extensions for some classes of linear differential operators in the Hilbert space of vector-functions at finite interval.

In 1985 by Pivtorak [26] and Ismailov [27] all solvable extensions of a minimal operator generated by linear parabolic and hyperbolic type differential expressions for first order with constant unbounded or continuously dependent selfadjoint operator coefficients in Hilbert space of vector-functions at finite interval in terms of boundary values were given, respectively.

In the studies discussed above the coefficients of differential expressions have been taken for special classes of operators in corresponding functional space. Unfortunately, in many cases representation of functional differential expression is not possible with remarkable coefficient, then mentioned above methods are not applicable to the study of these problems. On the other hand in noted above works spectral investigations have not been done.

Let us remember that an operator $T : D(T) \subset H \to H$ in Hilbert space H is called boundedly solvable, if T is one-to-one, $TD(T) = H$ and $T^{-1} \in L(H)$.

In this work, in Sect. 2, by using methods of operator theory all boundedly by solvable extensions of minimal operator generated by some functional differential-operator expression for first order in the Hilbert space of vector-functions at a finite interval have been described in terms of boundary values. Structure of the spectrums of these extensions has been given in Sect. 3. Lastly, in Sect. 4 the obtained results have been supported by applications.

2 Description of boundedly solvable extensions

In the Hilbert space $L^2(H, (0, 1))$ of vector-functions consider a linear functional differential-operator expression for first order in the form

$$l(u) = u'(t) + \sum_{m=1}^{n} A_m(t)u\left(\alpha_m|t - \lambda_m|^{\gamma_m}\right), \tag{1}$$

where:

(1) H is a separable Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$;
(2) For each $m,\ 1 \le m \le n$ operator-function $A_m(\cdot) : [0, 1] \to L(H)$ is continuous on the uniformly operator topology;
(3) For $m = 1, 2, \ldots, n,\ 0 < \alpha_m \le 1,\ 0 \le \lambda_m \le 1$ and $0 < \gamma_m \le 1$.

On the other hand, the following simple differential expression will be considered

$$m(u) = u'(t) \tag{2}$$

in the Hilbert space $L^2(H, (0, 1))$ corresponding to (1).

By the standard way the minimal M_0 and maximal M operators corresponding to (2).

Now define an operator $P(\alpha_m, \lambda_m, \gamma_m)$ in $L^2(H, (0, 1))$ in a form

$$P(\alpha_m, \lambda_m, \gamma_m)u(t) = u\left(\alpha_m|t - \lambda_m|^{\gamma_m}\right),\ u \in L^2(H, (0, 1)),\ m = 1, 2, \ldots, n$$

Then for $u \in L^2(H, (0, 1))$ and for $m = 1, 2, \ldots, n$ it is obtained that

$$\begin{aligned}\|P(\alpha_m, \lambda_m, \gamma_m)u\|^2_{L^2(H,(0,1))} &= \int_0^1 \|u\left(\alpha_m|t - \lambda_m|^{\gamma_m}\right)\|^2_H dt \\ &= \int_0^{\lambda_m} \|u\left(\alpha_m|t - \lambda_m|^{\gamma_m}\right)\|^2_H dt\end{aligned}$$

$$+\int_{\lambda_m}^{1}\|u\left(\alpha_m|t-\lambda_m|^{\gamma_m}\right)\|_H^2dt$$

$$=\int_{0}^{\lambda_m}\|u\left(\alpha_m\left(\lambda_m-t\right)^{\gamma_m}\right)\|_H^2dt$$

$$+\int_{\lambda_m}^{1}\|u\left(\alpha_m|t-\lambda_m|^{\gamma_m}\right)\|_H^2dt$$

$$=\frac{1}{\gamma_m}\left(\frac{1}{\alpha_m}\right)^{\frac{1}{\gamma_m}}\int_{0}^{\alpha_m(\lambda_m)^{\gamma_m}}\|u(x)\|_H^2x^{\frac{1-\gamma_m}{\gamma_m}}dx$$

$$+\frac{1}{\gamma_m}\left(\frac{1}{\alpha_m}\right)^{\frac{1}{\gamma_m}}\int_{0}^{\alpha_m(1-\lambda_m)^{\gamma_m}}\|u(x)\|_H^2x^{\frac{1-\gamma_m}{\gamma_m}}dx$$

$$\leq\frac{2}{\gamma_m}\left(\frac{1}{\alpha_m}\right)^{\frac{1}{\gamma_m}}\|u\|^2_{L^2(H,(0,1))}$$

So for each $m=1,2,\ldots,n$

$$P(\alpha_m,\lambda_m,\gamma_m)\in L(L^2(H,(0,1)))$$

and

$$\|P(\alpha_m,\lambda_m,\gamma_m)\|\leq\sqrt{\frac{2}{\gamma_m}}\left(\frac{1}{\alpha_m}\right)^{\frac{1}{2\gamma_m}}$$

Consequently, the operator

$$A(t;\alpha,\lambda,\gamma)=\sum_{m=1}^{n}A_m(t)P(\alpha_m,\lambda_m,\gamma_m)$$

is a linear bounded operator in $L^2(H,(0,1))$.

Along of this work the following defined operators

$$\begin{aligned}
&L_0:=M_0+A\left(t;\alpha,\lambda,\gamma\right),\\
&L_0:\overset{o}{W}{}_2^1(H,(0,1))\subset L^2(H,(0,1))\to L^2(H,(0,1))\\
&L:=M+A\left(t;\alpha,\lambda,\gamma\right),\\
&L:W_2^1(H,(0,1))\subset L^2(H,(0,1))\to L^2(H,(0,1))
\end{aligned}$$

will be called the minimal and maximal operators corresponding to differential expression (1) in $L^2(H,(0,1))$, respectively.

Now let $U(t,s),\ t,s \in [0,1]$ be the family of evolution operators corresponding to the homogeneous differential equation

$$\begin{cases} \frac{\partial U}{\partial t}(t,s)f + A(t;\alpha,\lambda,\gamma)U(t,s)f = 0,\ t,s \in [0,1], \\ U(s,s)f = f,\ f \in H \end{cases}$$

The operator $U(t,s),\ t,s \in [0,1]$ is a linear continuous, boundedly invertible in H and

$$U^{-1}(t,s) = U(s,t),\ s,t \in [0,1]$$

(for more detailed analysis of this concept see [28]).

Let us introduce the operator

$$Uz(t) := U(t,0)z(t),\ U : L^2(H,(0,1)) \to L^2(H,(0,1))$$

In this case it is easy to see that for the differentiable vector-function $z \in L^2(H,(0,1)),\ z : [0,1] \to H$ satisfies the following relation:

$$\begin{aligned} l(Uz) &= (Uz)'(t) + A(t;\alpha,\lambda,\gamma)Uz(t) \\ &= Uz'(t) + (U_t' + A(t;\alpha,\lambda,\gamma)U)z(t) = Um(z) \end{aligned}$$

From this $U^{-1}l(Uz) = m(z)$. Hence it is clear that if $\widetilde{L}$ is some extension of the minimal operator L_0, that is $L_0 \subset \widetilde{L} \subset$ L, then

$$U^{-1}L_0U = M_0,\ M_0 \subset U^{-1}LU = \widetilde{M} \subset M,\ U^{-1}LU = M$$

For example, prove the validity of the last relation. It is known that

$$D(M_0) = \overset{o}{W}{}_2^1(H,(0,1)),\ D(M) = W_2^1(H,(0,1)),$$

If $u \in D(M)$, then $l(Uz) = Um(z) \in L^2(H,(0,1))$ that is $Uu \in D(L)$. From the last relation $M \subset U^{-1}LU$. Contrarily, if a vector-function $u \in D(L)$, then

$$m(U^{-1}v) = U^{-1}l(v) \in L^2(H,(0,1)),$$

that is, $U^{-1}v \in D(M)$. From last relation $U^{-1}L \subset MU$, that is $U^{-1}LU \subset M$. Hence, $U^{-1}LU = M$.

It is easy to prove the following assertion.

Lemma 2.1 *$ker L_0 = \{0\}$ and $\overline{R(L_0)} \neq L^2(H,(0,1))$.*

Theorem 2.1 *Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H,(0,1))$ is generated by the functional differential-operator expression (1) and boundary condition*

$$(K+E)u(0)=KU(0,1)u(1), \tag{3}$$

where $K \in L(H)$ and E is an identity operator in H. The operator K is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L}=L_K$.

On the contrary, the restriction of the maximal operator L_0 to the manifold of vector-functions satisfies the condition (3) for some bounded operator $K \in L(H)$ is a solvable extension of the minimal operator L_0 in the $L^2(H,(0,1))$.

Proof Firstly, all solvable extensions $\widetilde{M}$ of the minimal operator M_0 in $L^2(H,(0,1))$ in terms of boundary values is described.

Consider the following so-called Cauchy extension M_c

$$M_c u = u'(t),\ u(0)=0,$$
$$M_c : D(M_c)=\{u \in W_2^1(H,(0,1)) : u(0)=0\} \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

of the minimal operator M_0. It is clear that M_c is a solvable extension of M_0 and

$$M_c^{-1}f(t)=\int_0^t f(x)dx,\ f \in L^2(H,(0,1)),$$
$$M_c^{-1} : L^2(H,(0,1)) \to L^2(H,(0,1))$$

Now assume that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2(H,(0,1))$. In this case it is known that the domain of $\widetilde{M}$ can be written in direct sum in form

$$D(\widetilde{M})=D(M_0)\oplus(M_c^{-1}+K)V,$$

where $V=ker M=H$, $K \in L(H)$ [21,22]. Therefore for each $u(t) \in D(\widetilde{M})$ it is true that

$$u(t)=u_0(t)+M_c^{-1}f+Kf, u_0 \in D(M_0),\ f \in H$$

That is $u(t)=u_0(t)+tf+Kf, u_0 \in D(M_0),\ f \in H$. Hence $u(0)=Kf$, $u(1)=f+Kf=(K+E)f$. Hence $u(0)=Kf$, $u(1)=f+Kf=(K+E)f$ and from these relations it is obtained that

$$(K+E)u(0)=Ku(1) \tag{4}$$

On the other hand, uniqueness of operator $K \in L(H)$ follows from [21]. Therefore, $\widetilde{M}=M_K$. This completes the necessary part of this assertion.

On the contrary, if M_K is an operator generated by differential expression (2) and boundary condition (4), then M_K is bounded, boundedly invertible and

$$M_K^{-1} : L^2(H, (0, 1)) \to L^2(H, (0, 1)),$$
$$M_K^{-1} f(t) = \int_0^t f(x)dx + K \int_0^1 f(x)dx, \; f \in L^2(H, (0, 1)).$$

Consequently, all solvable extensions of the minimal operator M_0 in $L^2(H, (0, 1))$ is generated by differential expression (2) and boundary condition (4) with any linear bounded operator K.

Now consider the general case. For this in the $L^2(H, (0, 1))$ introduce an operator in the form

$$U : L^2(H, (0, 1)) \to L^2(H, (0, 1)), \; (Uz)(t) := U(t, 0)z(t), z \in L^2(H, (0, 1))$$

From the properties of the family of evolution operators $U(t, s), t, s \in [0, 1]$ it is implied that an operator U is linear bounded, has a bounded inverse and

$$(U^{-1}z)(t) = U(0, t)z(t).$$

On the other hand, from the relations

$$U^{-1}L_0U = M_0, U^{-1}\widetilde{L}U = \widetilde{M}, U^{-1}LU = M \tag{5}$$

it is implied that an operator U is a one-to-one between sets of solvable extensions of minimal operators L_0 and M_0 in $L^2(H, (0, 1))$.

The extension $\widetilde{L}$ of the minimal operator L_0 is solvable in $L^2(H, (0, 1))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}U$ is an extension of the minimal M_0 in $L^2(H, (0, 1))$. Then, $u \in D(\widetilde{L})$ if and only if

$$(K + E)U(0, 0)u(0) = KU(0, 1)u(1),$$

that is, $(K + E)u(0) = KU(0, 1)u(1)$. This proves the validity of the claims in the theorem. □

Corollary 2.1 *In particular the resolvent operator $R_\lambda(L_K)$, $\lambda \in \rho(L_K)$ of any solvable extension L_K of the minimal operator L_0, generated by pantograph-type functional differential expression*

$$l(u) = u'(t) + A(t)u(\alpha t), \; 0 < \alpha < 1$$

with boundary condition

$$(K + E)u(0) = KU(0, 1)u(1),$$

in $L^2(H,(0,1))$ *is of the form*

$$R_\lambda(L_K)f(t) = U(t,0)[(E + K(1 - exp(\lambda))^{-1})K\int_0^1 exp(\lambda(1-s))U(0,s)f(s)ds$$
$$+\int_0^t exp(\lambda(1-s))U(0,s)f(s)ds], \quad f \in L^2(H,(0,1))$$

3 Spectrum of boundedly solvable extensions

In this section, the spectrum structure of solvable extensions of minimal operator L_0 in $L^2(H,(0,1))$ will be investigated.

Firstly, prove the following fact.

Theorem 3.1 *If* $\widetilde{L}$ *is a solvable extension of a minimal operator* L_0 *and* $\widetilde{M} = U^{-1}\widetilde{L}U$ *corresponds for the solvable extension of a minimal operator* M_0*, then the spectrum of these extensions is true* $\sigma(\widetilde{L}) = \sigma(\widetilde{M})$.

Proof Consider a problem to the spectrum for a solvable extension L_K of a minimal operator L_0 generated by functional differential-operator expression (1), that is

$$L_K u = \lambda u + f, \ \lambda \in \mathbb{C}, \ f \in L^2(H,(0,1))$$

From this it is obtained that

$$(L_K - \lambda E)u = f$$

or $(UM_KU^{-1} - \lambda E)u = f$. Hence $U(M_K - \lambda)(U^{-1}u) = f$.

Therefore, the validity of the theorem is clear.

Now prove the following result for the spectrum of solvable extension. □

Theorem 3.2 *If* L_K *a solvable extension of the minimal operator* L_0 *in the space* $L^2(H,(0,1))$*, then spectrum of* L_K *has the form*

$$\sigma(L_K) = \left\{\lambda \in \mathbb{C} : \lambda = \ln\left|\frac{\mu+1}{\mu}\right| + i\arg\left(\frac{\mu+1}{\mu}\right) + 2n\pi i, \mu \in \sigma(K)\backslash\{0,-1\}, \ n \in \mathbb{Z}\right\}$$

Proof Firstly, the spectrum of the solvable extension $M_K = U^{-1}L_KU$ of the minimal operator M_0 in $L^2(H,(0,1))$ will be investigated.

Consequently, consider the following problem for the spectrum, that is, $M_Ku = \lambda u + f$, $\lambda \in C$, $f \in L^2(H,(0,1))$. Then

$$u' = \lambda u + f,$$
$$(K + E)u(0) = Ku(1), \ \lambda \in C, \ f \in L^2(H,(0,1)), \ K \in L(H)$$

It is clear that a general solution of the above differential equation in $L^2(H, (0, 1))$ has the form

$$u_\lambda(t) = \exp(\lambda t) f_0 + \int_0^t \exp(\lambda(1-s)) f(s) ds, \ \ f_0 \in H$$

Therefore, from the boundary condition $(K + E)u_\lambda(0) = K u_\lambda(1)$ it is obtained that

$$(E + K(1 - \exp(\lambda))) f_0 = K \int_0^1 \exp(\lambda(1-s)) f(s) ds \tag{6}$$

For $\lambda_m = 2m\pi i, \ m \in \mathbb{Z}$ from the last relation it is established that

$$f_0^{(m)} = K \int_0^1 \exp(\lambda_m(1-s)) f(s) ds, \ m \in \mathbb{Z}$$

Consequently, in this case the resolvent operator of M_K is in the form

$$R_{\lambda_m}(M_K) f(t) = K \exp(\lambda_m t) \int_0^1 \exp(\lambda_m(1-s)) f(s) ds + \int_0^t \exp(\lambda(1-s)) f(s) ds, \ f \in L^2(H, (0, 1)), \ m \in \mathbb{Z}$$

On the other hand, it is clear that $R_{\lambda_m}(M_K) \in B((L^2(H, (0, 1)), m \in \mathbb{Z}$.

Now assume that $\lambda \neq 2m\pi i, \ m \in \mathbb{Z}, \ \lambda \in \mathbb{C}$. Then using the Eq. (6) we have

$$\left(K - \frac{1}{\exp(\lambda) - 1} E \right) f_0 = \frac{1}{1 - \exp(\lambda)} K \int_0^1 \exp(\lambda(1-s)) f(s) ds, \quad f_0 \in H, \ f \in L^2(H, (0, 1)$$

Therefore, for $\lambda \in \sigma(M_K)$ if and only if

$$\mu = \frac{1}{\exp(\lambda) - 1} \in \sigma(K)$$

In this case since $\mu \neq 0$

$$\exp(\lambda) = \frac{\mu + 1}{\mu}, \ \mu \in \sigma(K) \text{ and } \mu \neq -1$$

Then

$$\lambda_n = ln\left|\frac{\mu+1}{\mu}\right| + iarg\left(\frac{\mu+1}{\mu}\right) + 2n\pi i,\ n \in \mathbb{Z}$$

Later on, using the last relation and Theorem 3.1 the validity of the claim in theorem is proved. □

4 Applications

Example 4.1 Consider the following boundary value problem for the functional differential equation in form

$$\begin{cases} u'(t) = a(t)u(\sqrt{t}),\ 0 < t < 1,\ a \in C^1[0, 1] \\ u(0) = u_0 \end{cases}$$

In order to solve this problem change the unknown function $u(t)$ by

$$y(t) = u(t) - u_0,\ \ 0 < t < 1$$

Then we have

$$\begin{cases} y'(t) = a(t)y(\sqrt{t}) + a(t)u_0 \\ y(0) = 0 \end{cases}$$

The last boundary value problem can be interpretated as the solution of functional differential equation in Hilbert space $L^2(0, 1)$

$$\begin{cases} L_c y(t) = a(t)u_0,\ 0 < t < 1 \\ y(0) = 0 \end{cases}$$

where $L_c y(t) = y'(t) - a(t)y(\sqrt{t})$. Hence solution of above Cauchy problem by Corollary 2.1 can be analytically represented in the following form

$$y(t) = L_C^{-1}(a(t)u_0) = \int\limits_0^t U(0, s)a(s)dsu_0$$

Consequently,

$$u(t) = \int\limits_0^t U(0, s)a(s)dsu_0 + u_0,\ \ 0 < t < 1.$$

Here $U(t,s),\ t,s \in [0,1]$ is a family of evolution operators corresponding to problem

$$\begin{cases} \frac{\partial}{\partial t} U(t,s)f - a(t)P_\alpha U(t,s)f = 0, \\ U(s,s)f = f,\ f \in \mathbb{C} \end{cases}$$

with $P_\alpha u(t) = u(\sqrt{t}),\quad P_\alpha : L^2(0,1) \to L^2(0,1)$.

In general by Theorem 2.1 all boundedly solvable extensions L_k of the minimal operator L_0 generated by $l(u) = u'(t) - a(t)u(\sqrt{t}),\ 0 < t < 1$ in $L^2(0,1)$ are described $l(\cdot)$ with boundary condition

$$(k+1)u(0) = k\exp\left(\int_0^1 a(s)P_\alpha ds\right)u(1),\ k \in \mathbb{C}$$

In addition, the resolvent operator of these extensions is in the form

$$R_\lambda(L_k)f(t) = \exp\left(\int_0^t a(s)P_\alpha ds\right)\Bigg[(1+k(1-e^\lambda)^{-1})k \int_0^1 \exp\left(\lambda(1-s) - \int_0^s a(x)P_\alpha dx\right)f(s)ds + \int_0^t \exp\left(\lambda(t-s) - \int_0^s a(x)P_\alpha dx\right)f(s)ds\Bigg],\ \lambda \in \rho(L_k),\ f \in L^2(0,1)$$

Moreover for $k \neq 0, -1,\ k \in \mathbb{C}$ spectrum of this extension L_k is in the form,

$$\sigma(L_k) = \left\{\lambda \in \mathbb{C} : \lambda = \ln\left|\frac{k+1}{k}\right| + i\arg\left(\frac{k+1}{k}\right) + 2n\pi i,\ n \in \mathbb{Z}\right\}.$$

Example 4.2 All boundedly solvable extensions of minimal operator generated by functional differential expression

$$l(u) = \frac{\partial u(t,x)}{\partial t} + x^2 u(\sqrt{t}, \sqrt{1-x}),\ 0 < t,\ x < 1$$

in the Hilbert space $L^2((0,1)\times(0,1))$ are described by this $l(\cdot)$ and boundary condition

$$(K+E)u(0,x) = KU(0,1)u(1,x),$$

where K is 2×2-matrix and $U(t,s), t,s \in [0,1]$ is an operator solution of equation

$$\begin{cases} \frac{\partial U}{\partial t}(t,s)f + x^2 P_1 P_2 U(t,s)f = 0,\ t,s \in [0,1], \\ U(s,s)f = f,\ f \in \mathbb{C} \end{cases}$$

where $P_1u(t,x) = u(\sqrt{t},x)$, $P_2u(t,x) = u(t,\sqrt{1-x})$, $P_1, P_2 : L^2((0,1)\times(0,1)) \to L^2((0,1)\times(0,1))$.

Remark 4.3 In special case of α_m, λ_m and γ_m, $m = 1, 2, \ldots, n$ the analogous problems considered in this work have been investigated in papers [29–31].

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